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Some sharp inequalities for multilinear integral operators

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Abstract

In this paper, some sharp inequalities for certain multilinear operators related to the Littlewood-Paley operator and the Marcinkiewicz operator are obtained. As an application, we obtain the (L^p, L^q)-norm inequalities and Morrey spaces boundedness for the multilinear operators.

MSC: 42B20; 42B25

Keywords: multilinear operator; Littlewood-Paley operator; Marcinkiewicz operator; Morrey space; BMO

1 Introduction and results

In this paper, we study some multilinear operators related to some integral operators, whose definitions are as follows.

Fix $n > \delta \geq 0$. We denote $\Gamma(x) = \{(y,t) \in R^{n+1}_+ : |x-y| < t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. Suppose that m_j are the positive integers $(j = 1, \ldots, l)$, $m_1 + \cdots + m_l = m$ and A_j are the functions on R^n $(j = 1, \ldots, l)$. Let

$$R_{m_j+1}(A_j; x, y) = A_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

Definition 1 Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x)\,dx = 0$,

(2) $|\psi(x)| \leq C(1+|x|)^{-(n+1-\delta)}$,

(3) $|\psi(x+y) - \psi(x)| \leq C|y|^\varepsilon (1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

The multilinear Littlewood-Paley operator is defined by

$$S^A_\psi(f)(x) = \left[\int \int_{\Gamma(x)} \left| F^A_t(f)(x,y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right]^{1/2},$$

where

$$F^A_t(f)(x,y) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, z)}{|x-z|^m} \psi_t(y-z) f(z)\,dz$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$S_\psi(f)(x) = \left(\int\int_{\Gamma(x)} \left|F_t(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2},$$

which is the Littlewood-Paley operator (see [1]).

Let H be the Hilbert space $H = \{h : \|h\| = (\int\int_{R_+^{n+1}} |h(y,t)|^2\,dy\,dt/t^{n+1})^{1/2} < \infty\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$S_\psi^A(f)(x) = \left\|\chi_{\Gamma(x)}F_t^A(f)(x,y)\right\|, \qquad S_\psi(f)(x) = \left\|\chi_{\Gamma(x)}F_t(f)(y)\right\|.$$

Definition 2 Let $0 < \gamma \le 1$ and Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$. Assume that $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$. The multilinear Marcinkiewicz operator is defined by

$$\mu_S^A(f)(x) = \left[\int\int_{\Gamma(x)} \left|F_t^A(f)(x,y)\right|^2 \frac{dy\,dt}{t^{n+3}}\right]^{1/2},$$

where

$$F_t^A(f)(x,y) = \int_{|y-z|\le t} \frac{\prod_{j=1}^l R_{m_j+1}(A_j;x,z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f(z)\,dz.$$

Set

$$F_t(f)(y) = \int_{|y-z|\le t} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f(z)\,dz.$$

We also define that

$$\mu_S(f)(x) = \left(\int\int_{\Gamma(x)} \left|F_t(f)(y)\right|^2 \frac{dy\,dt}{t^{n+3}}\right)^{1/2},$$

which is the Marcinkiewicz operator (see [2]).

Let H be the Hilbert space $H = \{h : \|h\| = (\int\int_{R_+^{n+1}} |h(y,t)|^2\,dy\,dt/t^{n+3})^{1/2} < \infty\}$, then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$\mu_S^A(f)(x) = \left\|\chi_{\Gamma(x)}F_t^A(f)(x,y)\right\|, \qquad \mu_S(f)(x) = \left\|\chi_{\Gamma(x)}F_t(f)(y)\right\|.$$

Note that when $m = 0$, S_ψ^A and μ_S^A are just the multilinear commutators (see [3, 4]). While when $m > 0$, S_ψ^A and μ_S^A are non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [5–9]). In [10], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [11–13], authors proved a sharp estimate for the multilinear commutator. The main purpose of this paper is to prove the sharp inequalities for the multilinear integral operators S_ψ^A and μ_S^A when $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$. As an application, we obtain the (L^p, L^q)-norm inequalities and Morrey spaces boundedness for the multilinear operators.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \left| f(y) - f_Q \right| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)\, dx$. It is well-known that (see [14, 15])

$$f^{\#}(x) = \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q \left| f(y) - c \right| dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and $\|f\|_{BMO} = \|f^{\#}\|_{L^\infty}$. For $1 \le p < \infty$ and $0 \le \delta < n$, let

$$M_{\delta,p}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-p\delta/n}} \int_Q \left| f(y) \right|^p dy \right)^{1/p};$$

we write that $M_\mu(f) = M_{n\mu,1}(f)$, which is the fractional maximal operator.

Fixed $\lambda > 0$. For $1 \le p < \infty$, let

$$\|f\|_{L^{p,\lambda}} = \sup_{x \in R^n, d > 0} \left(\frac{1}{d^\lambda} \int_{B(x,d)} \left| f(y) \right|^p dy \right)^{1/p},$$

where $B(x,d) = \{y \in R^n : |x - y| < d\}$. The Morrey spaces are defined by (see [16–20])

$$L^{p,\lambda}\left(R^n\right) = \left\{ f \in L^1_{loc}\left(R^n\right) : \|f\|_{L^{p,\lambda}} < \infty \right\}.$$

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the multilinear integral operator on the Morrey space.

We shall prove the following theorems.

Theorem 1 *Let $D^\alpha A_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$.*

(1) *Then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$, $1 < r < n/\delta$ and $x \in R^n$,*

$$\left(S_\psi^A(f)\right)^{\#}(x) \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} \left\| D^{\alpha_j} A_j \right\|_{BMO} \right) M_{\delta,r}(f)(x);$$

(2) *If $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$, then S_ψ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is,*

$$\left\| S_\psi^A(f) \right\|_{L^q} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} \left\| D^{\alpha_j} A_j \right\|_{BMO} \right) \|f\|_{L^p};$$

(3) *If* $1 < p < n/\delta$, $0 < \lambda < n - p\delta$, $1/q = 1/p - \delta/(n-\lambda)$, *then* S_ψ^A *is bounded from* $L^{p,\lambda}(R^n)$ *to* $L^{q,\lambda}(R^n)$, *that is*,

$$\left\|S_\psi^A(f)\right\|_{L^{q,\lambda}} \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{BMO}\right)\|f\|_{L^{p,\lambda}}.$$

Theorem 2 *Let* $D^\alpha A_j \in BMO(R^n)$ *for all* α *with* $|\alpha| = m_j$ *and* $j = 1, \ldots, l$.

(1) *Then there exists a constant* $C > 0$ *such that for any* $f \in C_0^\infty(R^n)$, $1 < r < n/\delta$ *and* $x \in R^n$,

$$\left(\mu_S^A(f)\right)^\#(x) \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{BMO}\right)M_{\delta,r}(f)(x);$$

(2) *If* $1 < p < n/\delta$ *and* $1/p - 1/q = \delta/n$, *then* μ_S^A *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$, *that is*,

$$\left\|\mu_S^A(f)\right\|_{L^q} \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{BMO}\right)\|f\|_{L^p};$$

(3) *If* $1 < p < n/\delta$, $0 < \lambda < n - p\delta$, $1/q = 1/p - \delta/(n-\lambda)$, *then* μ_S^A *is bounded from* $L^{p,\lambda}(R^n)$ *to* $L^{q,\lambda}(R^n)$, *that is*,

$$\left\|\mu_S^A(f)\right\|_{L^{q,\lambda}} \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}\left\|D^{\alpha_j}A_j\right\|_{BMO}\right)\|f\|_{L^{p,\lambda}}.$$

Remark The conclusions of Theorems 1 and 2 are completely the same. Thus, they explain that the Littlewood-Paley and Marcinkiewicz operators have the many similar bondedness properties.

2 Proofs of theorems

To prove the theorems, we need the following lemmas.

Lemma 1 [7] *Let A be a function on* R^n *and* $D^\alpha A \in L^q(R^n)$ *for all* α *with* $|\alpha| = m$ *and some* $q > n$. *Then*

$$\left|R_m(A;x,y)\right| \le C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\left|D^\alpha A(z)\right|^q dz\right)^{1/q},$$

where $\tilde{Q}$ *is the cube centered at x and having side length* $5\sqrt{n}|x-y|$.

Lemma 2 [21] *Suppose that* $1 \le r < p < n/\delta$ *and* $1/q = 1/p - \delta/n$. *Then*

$$\left\|M_{\delta,r}(f)\right\|_{L^q} \le C\|f\|_{L^p}.$$

Lemma 3 [16, 17] *Let* $1 < p < \infty$ *and* $0 < \lambda < n$. *Then the following estimates hold*:

(a) $\|M(f)\|_{L^{p,\lambda}} \le C\|f^\#\|_{L^{p,\lambda}}$;

(b) $\|M_\mu(f)\|_{L^{q,\lambda}} \le C\|f\|_{L^{p,\lambda}}$ *for* $0 < \mu < (n-\lambda)/np$ *and* $1/q = 1/p - n\eta/(n-\lambda)$.

Lemma 4 *Let $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then S_ψ and μ_S are all bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Proof For S_ψ, by Minkowski inequality and the condition of ψ, we have

$$\begin{aligned} S_\psi(f)(x) &\le \int_{R^n} |f(z)| \left(\int_{\Gamma(x)} \left|\psi_t(y-z)\right|^2 \frac{dy\,dt}{t^{1+n}} \right)^{1/2} dz \\ &\le C \int_{R^n} |f(z)| \left(\int_0^\infty \int_{|x-y|\le t} \frac{t^{-2n+2\delta}}{(1+|y-z|/t)^{2n+2-2\delta}} \frac{dy\,dt}{t^{1+n}} \right)^{1/2} dz \\ &\le C \int_{R^n} |f(z)| \left(\int_0^\infty \int_{|x-y|\le t} \frac{2^{2n+2} t^{1-n}}{(2t+|y-z|)^{2n+2-2\delta}} \, dy\,dt \right)^{1/2} dz, \end{aligned}$$

noting that $2t + |y-z| \ge 2t + |x-z| - |x-y| \ge t + |x-z|$ when $|x-y| \le t$ and

$$\int_0^\infty \frac{t\,dt}{(t+|x-z|)^{2n+2-2\delta}} = C|x-z|^{-2n+2\delta},$$

we obtain

$$\begin{aligned} S_\psi(f)(x) &\le C \int_{R^n} |f(z)| \left(\int_0^\infty \frac{t\,dt}{(t+|x-z|)^{2n+2-2\delta}} \right)^{1/2} dz \\ &= C \int_{R^n} \frac{|f(z)|}{|x-z|^{n-\delta}} \, dz. \end{aligned}$$

For μ_S, note that $|x-z| \le 2t$, $|y-z| \ge |x-z| - t \ge |x-z| - 3t$ when $|x-y| \le t$, $|y-z| \le t$, we have

$$\begin{aligned} \mu_S(f)(x) &\le \int_{R^n} \left[\int \int_{|x-y|\le t} \left(\frac{|\Omega(y-z)||f(z)|}{|y-z|^{n-1-\delta}} \right)^2 \chi_{\Gamma(z)}(y,t) \frac{dy\,dt}{t^{n+3}} \right]^{1/2} dz \\ &\le C \int_{R^n} |f(z)| \left[\int \int_{|x-y|\le t} \frac{\chi_{\Gamma(z)}(y,t) t^{-n-3}}{(|x-z|-3t)^{2n-2-2\delta}} \, dy\,dt \right]^{1/2} dz \\ &\le C \int_{R^n} \frac{|f(z)|}{|x-z|^{3/2}} \left[\int_{|x-z|/2}^\infty \frac{dt}{(|x-z|-3t)^{2n-2}} \right]^{1/2} dz \\ &\le C \int_{R^n} \frac{|f(z)|}{|x-z|^{n-\delta}} \, dz. \end{aligned}$$

Thus, the lemma follows from [21]. □

Proof of Theorem 1 (1) It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|} \int_Q \left| S_\psi^A(f)(x) - C_0 \right| dx \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} \left\| D^{\alpha_j} A_j \right\|_{BMO} \right) M_{\delta,r}(f)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!} (D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y)$ and

$D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}
F_t^A(f)(x,y) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,z)}{|x-z|^m} \psi_t(y-z)f(z)\,dz \\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j;x,z)}{|x-z|^m} \psi_t(y-z)f_2(z)\,dz \\
&\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,z)}{|x-z|^m} \psi_t(y-z)f_1(z)\,dz \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2;x,z)(x-z)^{\alpha_1}}{|x-z|^m} D^{\alpha_1}\tilde{A}_1(z)\psi_t(y-z)f_1(z)\,dz \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1;x,z)(x-z)^{\alpha_2}}{|x-z|^m} D^{\alpha_2}\tilde{A}_2(z)\psi_t(y-z)f_1(z)\,dz \\
&\quad + \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-z)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(z) D^{\alpha_2}\tilde{A}_2(z)}{|x-z|^m} \psi_t(y-z)f_1(z)\,dz,
\end{aligned}$$

then

$$\begin{aligned}
&\left|S_\psi^A(f)(x) - S_\psi^{\tilde{A}}(f_2)(x_0)\right| \\
&\quad = \left| \left\| \chi_{\Gamma(x)} F_t^A(f)(x,y) \right\| - \left\| \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0,y) \right\| \right| \\
&\quad \le \left\| \chi_{\Gamma(x)} F_t^A(f)(x,y) - \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0,y) \right\| \\
&\quad \le \left\| \chi_{\Gamma(x)} \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,z)}{|x-z|^m} \psi_t(y-z)f_1(z)\,dz \right\| \\
&\qquad + \left\| \chi_{\Gamma(x)} \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2;x,z)(x-z)^{\alpha_1}}{|x-z|^m} D^{\alpha_1}\tilde{A}_1(z)\psi_t(y-z)f_1(z)\,dz \right\| \\
&\qquad + \left\| \chi_{\Gamma(x)} \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1;x,z)(x-z)^{\alpha_2}}{|x-z|^m} D^{\alpha_2}\tilde{A}_2(z)\psi_t(y-z)f_1(z)\,dz \right\| \\
&\qquad + \left\| \chi_{\Gamma(x)} \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-z)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(z) D^{\alpha_2}\tilde{A}_2(z)}{|x-z|^m} \psi_t(y-z)f_1(z)\,dz \right\| \\
&\qquad + \left\| \chi_{\Gamma(x)} F_t^{\tilde{A}}(f_2)(x,y) - \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0,y) \right\| \\
&\quad := I_1(x) + I_2(x) + I_3(x) + I_4(x) + I_5(x),
\end{aligned}$$

thus,

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q \left|S_\psi^A(f)(x) - S_\psi^{\tilde{A}}(f_2)(x_0)\right| dx \\
&\quad \le \frac{1}{|Q|} \int_Q I_1(x)\,dx + \frac{C}{|Q|} \int_Q I_2(x)\,dx + \frac{C}{|Q|} \int_Q I_3(x)\,dx \\
&\qquad + \frac{C}{|Q|} \int_Q I_4(x)\,dx + \frac{1}{|Q|} \int_Q I_5(x)\,dx \\
&\quad := I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x \in Q$ and $z \in \tilde{Q}$, by Lemma 1, we get

$$R_{m_j}(\tilde{A}_j; x, z) \le C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO}.$$

Thus, by the (L^r, L^q)-boundedness of S_ψ, for $1 < r < n/\delta$ and $1/q = 1/r - \delta/n$, we obtain

$$\begin{aligned}
I_1 &\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \frac{1}{|Q|} \int_Q \left| S_\psi(f_1)(x) \right| dx \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) \left(\frac{1}{|Q|} \int_Q |S_\psi(f_1)(x)|^q \, dx \right)^{1/q} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) |Q|^{-1/q} \left(\int_Q \left| f_1(x) \right|^r dx \right)^{1/r} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For I_2, denoting $r = pq$ for $1 < p < n/\delta$, $q > 1$, $1/q + 1/q' = 1$ and $1/s = 1/p - \delta/n$, we have, by Hölder's inequality,

$$\begin{aligned}
I_2 &\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_Q \left| S_\psi\left(D^{\alpha_1} \tilde{A}_1 f_1\right)(x) \right| dx \\
&\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} \left| S_\psi\left(D^{\alpha_1} \tilde{A}_1 f_1\right)(x) \right|^s dx \right)^{1/s} \\
&\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \sum_{|\alpha_1|=m_1} |Q|^{-1/s} \left(\int_{R^n} \left| D^{\alpha_1} \tilde{A}_1(x) f_1(x) \right|^p dx \right)^{1/p} \\
&\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{BMO} \\
&\quad \times \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{\tilde{Q}} \left| D^{\alpha_1} \tilde{A}_1(x) \right|^{pq'} dx \right)^{1/pq'} \left(\frac{1}{|Q|^{1-r\delta/n}} \int_{\tilde{Q}} \left| f(x) \right|^{pq} dx \right)^{1/pq} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^{\alpha} A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$I_3 \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^{\alpha} A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).$$

Similarly, for I_4, denoting $r = pq_3$ for $1 < p < n/\delta$, $q_1, q_2, q_3 > 1$, $1/q_1 + 1/q_2 + 1/q_3 = 1$ and $1/s = 1/p - \delta/n$, we obtain

$$\begin{aligned}
I_4 &\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{|Q|} \int_Q \left| S_\psi \left(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 f_1 \right)(x) \right| dx \\
&\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} \left| S_\psi \left(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 f_1 \right)(x) \right|^s dx \right)^{1/s} \\
&\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} |Q|^{-1/s} \left(\int_{R^n} \left| D^{\alpha_1} \tilde{A}_1(x) D^{\alpha_2} \tilde{A}_2(x) f_1(x) \right|^p dx \right)^{1/p} \\
&\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{\tilde{Q}} \left| D^{\alpha_1} \tilde{A}_1(x) \right|^{pq_1} dx \right)^{1/pq_1} \left(\frac{1}{|Q|} \int_{\tilde{Q}} \left| D^{\alpha_2} \tilde{A}_2(x) \right|^{pq_2} dx \right)^{1/pq_2} \\
&\quad \times \left(\frac{1}{|Q|^{1-r\delta/n}} \int_{\tilde{Q}} \left| f(x) \right|^{pq_3} dx \right)^{1/pq_3} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \left\| D^\alpha A_j \right\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$\begin{aligned}
&\chi_{\Gamma(x)} F_t^{\tilde{A}}(f_2)(x,y) - \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0,y) \\
&= \int_{R^n} \left(\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)} \right) \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, z)}{|x-z|^m} \psi_t(y-z) f_2(z)\, dz \\
&\quad + \chi_{\Gamma(x_0)} \int_{R^n} \left(\frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, z) \psi_t(y-z) f_2(z)\, dz \\
&\quad + \chi_{\Gamma(x_0)} \int_{R^n} \left(R_{m_1}(\tilde{A}_1; x, z) - R_{m_1}(\tilde{A}_1; x_0, z) \right) \frac{R_{m_2}(\tilde{A}_2; x, z)}{|x_0-z|^m} \psi_t(y-z) f_2(z)\, dz \\
&\quad + \chi_{\Gamma(x_0)} \int_{R^n} \left(R_{m_2}(\tilde{A}_2; x, z) - R_{m_2}(\tilde{A}_2; x_0, z) \right) \frac{R_{m_1}(\tilde{A}_1; x_0, z)}{|x_0-z|^m} \psi_t(y-z) f_2(z)\, dz \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{A}_2; x, z)(x-z)^{\alpha_1} \chi_{\Gamma(x)}}{|x-z|^m} - \frac{R_{m_2}(\tilde{A}_2; x_0, z)(x_0-z)^{\alpha_1} \chi_{\Gamma(x_0)}}{|x_0-z|^m} \right] \\
&\quad \times D^{\alpha_1} \tilde{A}_1(z) \psi_t(y-z) f_2(z)\, dz \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{A}_1; x, z)(x-z)^{\alpha_2} \chi_{\Gamma(x)}}{|x-z|^m} - \frac{R_{m_1}(\tilde{A}_1; x_0, z)(x_0-z)^{\alpha_2} \chi_{\Gamma(x_0)}}{|x_0-z|^m} \right] \\
&\quad \times D^{\alpha_2} \tilde{A}_2(z) \psi_t(y-z) f_2(z)\, dz \\
&\quad + \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \int_{R^n} \left[\frac{(x-z)^{\alpha_1+\alpha_2} \chi_{\Gamma(x)}}{|x-z|^m} - \frac{(x_0-z)^{\alpha_1+\alpha_2} \chi_{\Gamma(x_0)}}{|x_0-z|^m} \right] \\
&\quad \times D^{\alpha_1} \tilde{A}_1(z) D^{\alpha_2} \tilde{A}_2(z) \psi_t(y-z) f_2(z)\, dz \\
&= I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)} + I_5^{(7)}.
\end{aligned}$$

By Lemma 1 and the following inequality (see [15])

$$|b_{Q_1} - b_{Q_2}| \le C\log(|Q_2|/|Q_1|)\|b\|_{BMO} \quad \text{for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $z \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$\begin{aligned}\left|R_m(\tilde{A};x,z)\right| &\le C|x-z|^m \sum_{|\alpha|=m} \left(\|D^\alpha A\|_{BMO} + \left|(D^\alpha A)_{\tilde{Q}(x,z)} - (D^\alpha A)_{\tilde{Q}}\right|\right) \\ &\le Ck|x-z|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}.\end{aligned}$$

Note that $|x-z| \sim |x_0 - z|$ for $x \in Q$ and $z \in R^n \setminus \tilde{Q}$, we obtain, similar to the proof of Lemma 4,

$$\begin{aligned}\|I_5^{(1)}\| &\le \int_{R^n} \left(\int\int_{R_+^{n+1}} \left[\frac{\prod_{j=1}^2 |R_{m_j}(\tilde{A}_j;x,z)||\psi_t(y-z)||f_2(z)|}{|x-z|^m}\right.\right. \\ &\quad \left.\left. \times \left|\chi_{\Gamma(x)}(y,t) - \chi_{\Gamma(x_0)}(y,t)\right|\right]^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2} dz \\ &\le C\int_{R^n} \frac{\prod_{j=1}^2 |R_{m_j}(\tilde{A}_j;x,z)||f_2(z)|}{|x_0-z|^m} \\ &\quad \times \left|\int_{\Gamma(x)} \frac{t^{1-n}dy\,dt}{(t+|y-z|)^{2n+2-2\delta}} - \int_{\Gamma(x_0)} \frac{t^{1-n}dy\,dt}{(t+|y-z|)^{2n+2-2\delta}}\right|^{1/2} dz \\ &\le C\int_{R^n} \frac{\prod_{j=1}^2 |R_{m_j}(\tilde{A}_j;x,z)||f_2(z)|}{|x_0-z|^m} \\ &\quad \times \left(\int\int_{|y|\le t} \left|\frac{1}{(t+|x+y-z|)^{2n+2-2\delta}} - \frac{1}{(t+|x_0+y-z|)^{2n+2-2\delta}}\right| \frac{dy\,dt}{t^{n-1}}\right)^{1/2} dz \\ &\le C\int_{R^n} \frac{\prod_{j=1}^2 |R_{m_j}(\tilde{A}_j;x,z)||f_2(z)|}{|x_0-z|^m} \left(\int\int_{|y|\le t} \frac{|x-x_0|t^{1-n}dy\,dt}{(t+|x+y-z|)^{2n+3-2\delta}}\right)^{1/2} dz \\ &\le C\int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,z)|f_2(z)||x-x_0|^{1/2}}{|x_0-z|^{m+n+1/2-\delta}} dz \\ &\le C\prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO}\right) \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 \frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2-\delta}} \left|f(z)\right| dz \\ &\le C\prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO}\right) \sum_{k=1}^\infty k^2 2^{-k/2} \frac{1}{|2^k\tilde{Q}|^{1-\delta/n}} \int_{2^k\tilde{Q}} \left|f(z)\right| dz \\ &\le C\prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO}\right) M_{\delta,r}(f)(\tilde{x});\end{aligned}$$

$$\begin{aligned}\|I_5^{(2)}\| &\le C\int_{R^n} \frac{|x-x_0|}{|x_0-z|^{m+n+1-\delta}} \prod_{j=1}^2 \left|R_{m_j}(\tilde{A}_j;x,z)\right| \left|f_2(z)\right| dz \\ &\le C\prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO}\right) \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 \frac{|x-x_0|}{|x_0-z|^{n+1-\delta}} \left|f(z)\right| dz\end{aligned}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{-k}\frac{1}{|2^k\tilde{Q}|^{1-\delta/n}}\int_{2^k\tilde{Q}}|f(z)|\,dz$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

For $I_5^{(3)}$ and $I_5^{(4)}$, by the formula (see [7])

$$R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}\left(D^\beta\tilde{A};x,x_0\right)(x-z)^\beta$$

and Lemma 1, we have

$$\left|R_m(\tilde{A};x,z)-R_m(\tilde{A};x_0,z)\right|\leq C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-z|^{|\beta|}\|D^\alpha A\|_{BMO}.$$

Thus, similar to the proof of Lemma 4,

$$\|I_5^{(3)}\|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-z|^{n+1-\delta}}|f(y)|\,dy$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)M_{\delta,r}(f)(\tilde{x});$$

$$\|I_5^{(4)}\|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

Similarly, we get

$$\|I_5^{(5)}\|\leq C\sum_{|\alpha_1|=m_1}\int_{R^n}\left\|\left[\frac{R_{m_2}(\tilde{A}_2;x,z)(x-z)^{\alpha_1}\chi_{\Gamma(x)}}{|x-z|^m}-\frac{R_{m_2}(\tilde{A}_2;x_0,z)(x_0-z)^{\alpha_1}\chi_{\Gamma(x_0)}}{|x_0-z|^m}\right]\right.$$

$$\left.\times\,\psi_t(y-z)\right\|\left|D^{\alpha_1}\tilde{A}_1(z)\right|\left|f_2(z)\right|\,dz$$

$$\leq C\sum_{|\alpha|=m_2}\|D^\alpha A_2\|_{BMO}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k\left(2^{-k/2}+2^{-k}\right)$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\left|D^{\alpha_1}\tilde{A}_1(y)\right|^{r'}dy\right)^{1/r'}\left(\frac{1}{|2^k\tilde{Q}|^{1-r\delta/n}}\int_{2^k\tilde{Q}}|f(y)|^r\,dy\right)^{1/r}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)M_{\delta,r}(f)(\tilde{x});$$

$$\|I_5^{(6)}\|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

For $I_5^{(7)}$, taking $q_1, q_2 > 1$ such that $1/r + 1/q_1 + 1/q_2 = 1$, then

$$\begin{aligned}
\|I_5^{(7)}\| &\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \int_{R^n} \left\| \left[\frac{(x-z)^{\alpha_1+\alpha_2} \chi_{\Gamma(x)}}{|x-z|^m} - \frac{(x_0-z)^{\alpha_1+\alpha_2} \chi_{\Gamma(x_0)}}{|x_0-z|^m} \right] \psi_t(y-z) \right\| \\
&\quad \times \left|D^{\alpha_1}\tilde{A}_1(z)\right| \left|D^{\alpha_2}\tilde{A}_2(z)\right| \left|f_2(z)\right| dz \\
&\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \sum_{k=1}^{\infty} k\left(2^{-k/2}+2^{-k}\right) \left(\frac{1}{|2^k\tilde{Q}|^{1-p\delta/n}} \int_{2^k\tilde{Q}} \left|f(y)\right|^r dy \right)^{1/r} \\
&\quad \times \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \left|D^{\alpha_1}\tilde{A}_1(y)\right|^{q_1} dy \right)^{1/q_1} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \left|D^{\alpha_2}\tilde{A}_2(y)\right|^{q_2} dy \right)^{1/q_2} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).
\end{aligned}$$

Thus

$$\|I_5\| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) M_{\delta,r}(f)(\tilde{x}).$$

We choose $1 < r < p$ in (1), then (2) follows from Lemma 2. For (3), taking $1 < r < \min(p, (n-\lambda)/p\delta)$ in (1) and by Lemma 3, we obtain

$$\begin{aligned}
\|S_\psi^A(f)\|_{L^{q,\lambda}} &\le C\|M(S_\psi^A(f))\|_{L^{q,\lambda}} \le C\|(S_\psi^A(f))^\#\|_{L^{q,\lambda}} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \|M_{\delta,r}(f)\|_{L^{q,\lambda}} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \left\| \left(M_{r\delta/n}(|f|^r)\right)^{1/r} \right\|_{L^{q,\lambda}} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \left\| M_{r\delta/n}(|f|^r) \right\|_{L^{q/r,\lambda}}^{1/r} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \left\| |f|^r \right\|_{L^{p/r,\lambda}}^{1/r} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{BMO} \right) \|f\|_{L^{p,\lambda}}.
\end{aligned}$$

This completes the proof of Theorem 1. □

Proof of Theorem 2 It is only to prove (1). Let Q, $\tilde{Q}$, $\tilde{A}_j(x)$, f_1 and f_2 be the same as the proof of Theorem 1. We write

$$\begin{aligned}
&F_t^A(f)(x,y) \\
&\quad = \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_2(z)\, dz
\end{aligned}$$

$$+ \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_1(z)\,dz$$
$$- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, z)(x-z)^{\alpha_1} D^{\alpha_1}\tilde{A}_1(z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_1(z)\,dz$$
$$- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, z)(x-z)^{\alpha_2} D^{\alpha_2}\tilde{A}_2(z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_1(z)\,dz$$
$$+ \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-z)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(z) D^{\alpha_2}\tilde{A}_2(z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_1(z)\,dz,$$

then

$$\begin{aligned}
&\frac{1}{|Q|} \int_Q \bigl|\mu_S^A(f)(x) - \mu_S^{\tilde{A}}(f_2)(x_0)\bigr|\,dx \\
&\quad\le \frac{1}{|Q|} \int_Q \bigl\| \chi_{\Gamma(x)} F_t^A(f)(x,y) - \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0,y) \bigr\|\,dx \\
&\quad\le \frac{1}{|Q|} \int_Q \biggl\| \chi_{\Gamma(x)} \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_1(z)\,dz \biggr\|\,dx \\
&\qquad + \frac{1}{|Q|} \int_Q \biggl\| \chi_{\Gamma(x)} \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \\
&\qquad \times \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, z)(x-z)^{\alpha_1} D^{\alpha_1}\tilde{A}_1(z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_1(z)\,dz \biggr\|\,dx \\
&\qquad + \frac{1}{|Q|} \int_Q \biggl\| \chi_{\Gamma(x)} \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \\
&\qquad \times \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, z)(x-z)^{\alpha_2} D^{\alpha_2}\tilde{A}_2(z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_1(z)\,dz \biggr\|\,dx \\
&\qquad + \frac{1}{|Q|} \int_Q \biggl\| \chi_{\Gamma(x)} \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \\
&\qquad \times \int_{R^n} \frac{(x-z)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(z) D^{\alpha_2}\tilde{A}_2(z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_1(z)\,dz \biggr\|\,dx \\
&\qquad + \frac{1}{|Q|} \int_Q \bigl\| \chi_{\Gamma(x)} F_t^{\tilde{A}}(f_2)(x,y) - \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0,y) \bigr\|\,dx \\
&\quad := J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

Similar to the proof of Theorem 1, we get

$$\begin{aligned}
J_1 &\le C \prod_{j=1}^{2} \biggl(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \biggr) \frac{1}{|Q|} \int_Q \bigl|\mu_S(f_1)(x)\bigr|\,dx \\
&\le C \prod_{j=1}^{2} \biggl(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \biggr) \biggl(\frac{1}{|Q|} \int_Q \bigl|\mu_S(f_1)(x)\bigr|^q\,dx \biggr)^{1/q} \\
&\le C \prod_{j=1}^{2} \biggl(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{BMO} \biggr) M_{\delta,r}(f)(\tilde{x});
\end{aligned}$$

$$\begin{aligned}
J_2 &\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2}A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|}\int_Q \big|\mu_S\big(D^{\alpha_1}\tilde{A}_1 f_1\big)(x)\big|\,dx \\
&\le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2}A_2\|_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|}\int_{R^n} \big|\mu_S\big(D^{\alpha_1}\tilde{A}_1 f_1\big)(x)\big|^s\,dx\right)^{1/s} \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^{\alpha}A_j\|_{BMO}\right) M_{\delta,r}(f)(\tilde{x}); \\
J_3 &\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^{\alpha}A_j\|_{BMO}\right) M_{\delta,r}(f)(\tilde{x}); \\
J_4 &\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{|Q|}\int_Q \big|\mu_S\big(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 f_1\big)(x)\big|\,dx \\
&\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|Q|}\int_{R^n} \big|\mu_S\big(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 f_1\big)(x)\big|^s\,dx\right)^{1/s} \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^{\alpha}A_j\|_{BMO}\right) M_{\delta,p}(f)(\tilde{x}).
\end{aligned}$$

For J_5, we write

$$\begin{aligned}
&\chi_{\Gamma(x)} F_t^{\tilde{A}}(f_2)(x,y) - \chi_{\Gamma(x_0)} F_t^{\tilde{A}}(f_2)(x_0,y) \\
&\quad= \int_{R^n} (\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)}) \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,z)}{|x-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_2(z)\,dz \\
&\qquad+ \chi_{\Gamma(x_0)} \int_{R^n} \left(\frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m}\right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j;x,z) \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_2(z)\,dz \\
&\qquad+ \chi_{\Gamma(x_0)} \int_{R^n} \big(R_{m_1}(\tilde{A}_1;x,z) - R_{m_1}(\tilde{A}_1;x_0,z)\big) \frac{R_{m_2}(\tilde{A}_2;x,z)}{|x_0-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_2(z)\,dz \\
&\qquad+ \chi_{\Gamma(x_0)} \int_{R^n} \big(R_{m_2}(\tilde{A}_2;x,z) - R_{m_2}(\tilde{A}_2;x_0,z)\big) \frac{R_{m_1}(\tilde{A}_1;x_0,z)}{|x_0-z|^m} \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} f_2(z)\,dz \\
&\qquad- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{A}_2;x,z)(x-z)^{\alpha_1}\chi_{\Gamma(x)}}{|x-z|^m} - \frac{R_{m_2}(\tilde{A}_2;x_0,z)(x_0-z)^{\alpha_1}\chi_{\Gamma(x_0)}}{|x_0-z|^m}\right] \\
&\qquad\times \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} D^{\alpha_1}\tilde{A}_1(z) f_2(z)\,dz \\
&\qquad- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{A}_1;x,z)(x-z)^{\alpha_2}\chi_{\Gamma(x)}}{|x-z|^m} - \frac{R_{m_1}(\tilde{A}_1;x_0,z)(x_0-z)^{\alpha_2}\chi_{\Gamma(x_0)}}{|x_0-z|^m}\right] \\
&\qquad\times \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} D^{\alpha_2}\tilde{A}_2(z) f_2(z)\,dz \\
&\qquad+ \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \left[\frac{(x-z)^{\alpha_1+\alpha_2}\chi_{\Gamma(x)}}{|x-z|^m} - \frac{(x_0-z)^{\alpha_1+\alpha_2}\chi_{\Gamma(x_0)}}{|x_0-z|^m}\right] \\
&\qquad\times \frac{\Omega(y-z)}{|y-z|^{n-1-\delta}} D^{\alpha_1}\tilde{A}_1(z) D^{\alpha_2}\tilde{A}_2(z) f_2(z)\,dz.
\end{aligned}$$

Then, similar to the proof of Lemma 4 and Theorem 1, we get

$$\|J_5\| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{BMO}\right)M_{\delta,r}(f)(\tilde{x}).$$

The same argument as the proof of Theorem 1 will give the proof of (2) and (3), we omit the details and finish the proof. □

Competing interests
The author declares that he has no competing interests.

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